ON TREE ROOTS OF GRAPHS

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A graph $G' = (V, E')$ is defined to be the nth power of a graph $G = (V, E)$ if $E' = \{\{x, y\} \mid d(x, y) \leq n \text{ in } G\}$. G is said to be an nth root of G'. Every graph G has a unique nth power for all $n \geq 1$, but a graph may have zero or more nth roots. In this paper, we endeavour to devise an algorithm to determine whether a graph is some power of a tree T. Also, we assume that the given graph $G \neq K_p$, since in that case it is the nth power of all trees with same number of vertices and diameter $d \leq n/2$. Moreover, some of the lemmas assume that $d(G) > n$.

Keywords: Power of a graph, roots of a graph, powers of trees

1 Concepts, terms and definitions

We define below some terms which are used later in the paper. Graph theoretic definitions not given here are used in the sense of Harary[1]. Some more definitions and concepts will be introduced when required.

Definitions-

End-deleted tree - The end deleted tree T' of a given tree T is defined to be the tree obtained by deletion of the leaf nodes of T.

In the trivial case when T is a single node or K_2, T' is defined to be an empty graph.

ith order leaf nodes - The set of ith order leaf nodes L_i is defined to be that set of vertices which are leaf nodes of i-times end deleted tree of T. i.e., a vertex $v \in L_i$ if v is a leaf node in i-times end deleted tree. In the base case, L_0 is the set of leaf nodes of T.

In figure 1,

L_0={1,7,8,9,11}
L_1={2,6,10}
L_2={3,5}
L_3={4}
L_4={}

Distance between two edges - If e_1 and e_2 are two edges in the graph then the distance $d(e_1, e_2)$ between them is defined to be $n + 1$, where n is the number of edges on the shortest path between them.

Distance between an edge and a vertex - If $\{u_1, u_2\}$ is an edge e and v is a vertex in a graph G, then the distance $d(e, v)$ between e and v is defined as $d(e, v) = min\{d(u_1, v), d(u_2, v)\}$.

Span of a vertex - Let k be a natural number. If v is a vertex then the k-span of vertex v, $S(v, k)$ is defined to be the set of all vertices at a distance upto k from v, i.e.,
$S(v, k) = \{u|d(u, v) \leq k\}$
In particular, S(v,0) = $\{v\}$ and $|S(v, 1)|$ = deg(v) + 1

Span of an edge - Let k be natural number. If $e = \{v_1, v_2\}$ is an edge, then the k-span of edge e, $S(e, k)$ is defined to be the union of k-spans of its end-points, i.e.,

$$k\text{-span of } e = k\text{-span of } v_1 \cup\ k\text{-span of } v_2. \tag{1}$$

Lemma 1.1 - *For i and j such that $i \neq j$, $L_i \bigcap L_j = \phi$*

Proof: Assume without loss of generality, $i < j$.
If $v \in L_i$ then v is a leaf node in i-times end deleted tree of T.
$\therefore$ in subsequent end deleted trees, v is absent.
$\Rightarrow v \notin L_j$, where $j > i$.
$\therefore L_i \bigcap L_j = \phi$.

Lemma 1.2 - *If T is a tree of diameter d and $k = \lfloor d/2 \rfloor$ then $L_{k+i} = \phi$, $\forall i \geq 1$.*

Proof: A tree has either one center or two adjacent centers (Harary[1]). So the maximum distance between an end-point and a center is $\lfloor d/2 \rfloor + 1$. So after $\lfloor d/2 \rfloor + 1$ end-deletions, there will be no end points left at all.
$\therefore L_{k+i} = \phi\ \forall i \geq 1$.

Lemma 1.3 - $\bigcup_{1 \leq i \leq k} L_i = V$.

Proof: For $i > k$, $L_i = \phi$.
For $1 \leq i \leq k$, at each stage the end-points of that stage are deleted. Eventually since we have an empty set, it means that all points have been end-points at some stage of deletion.
$\therefore \bigcup_{1 \leq i \leq k} L_i = V$.

2 Clique Structure in Graphs Isomorphic to T^n

Let G be a graph such that $G \equiv T^n$ for some tree T. Our objective is to study the properties of the cliques of G.

If T is a tree, then a look at T^2 and T^3 yields the following observations:

(i) In T^2, the cliques are formed by all vertices of T which are not leaf nodes.

(ii) In T^3, the cliques are formed by all edges of T that are not connected to leaf nodes.

2.1 Characterization of Cliques

The clique structure of powers of a tree exhibits interesting structures. We characterize the power of a tree on the basis of its cliques.

Let n be an odd natural number.

Let $k = \frac{n-1}{2}$ and let $e = \{v_1, v_2\}$ be an edge. If v_1 and v_2 are such that $v_1, v_2 \in L_{k+i}$, $i \geq 0$ then the subgraph which is the k-span of e is a clique.

Such an edge is defined as a *clique edge.*

Now suppose n is even.

Let $k = \frac{n}{2}$ and let v be a vertex such that $v \in L_{k+i}$, $i \geq 0$. Then the subgraph which is the k-span of v is a clique.

Such a vertex v is defined as a *clique vertex.*

Lemma 2.1 - *If T is a tree and n is a positive integer such that $G \equiv T^n$, then the graph G has cliques only of the above characterstics.*

Proof: Suppose S is a clique in G where $G \equiv T^n$.
By definition, a clique is a maximal complete subgraph of a graph. Also since it is assumed that $G \neq K_p$,
$\therefore \exists u, v \in S$ such that $d(u, v) = n$ in T.
The following possibilities arise:
Case I: n **is odd**
$\therefore \exists$ an edge $e \ni d(e, u) = k$ and $d(e, v) = k$ where $k = \frac{n-1}{2}$.
Claim: S is the k-span of e.
Proof: $\forall w \in S$, we have in T,

$d(w, u) \leq n$ and $d(w, v) \leq n$.

In a tree the only path between u and v must have edge e.
$\therefore$ Either the path from w to u has e or the path from w to v has e (otherwise we will get a cycle in a tree).
$\therefore$ Either $d(w, u) = k + d(e, w) + 1$
or $d(w, v) = k + d(e, w) + 1$
$\therefore d(e, w) \leq k$

$\Rightarrow S \subseteq k-$span of e.
Also, $\forall w \in k$-span of e,
$d(w,v) \leq d(w,e) + d(e,v) + 1 \leq k + k + 1 = n, \forall v \in S$.
$\therefore$ k-span of $e \subseteq S$.
$\therefore$ For odd n, $S = k$-span of e.

Case II- n **is even**
$\therefore \exists$ vertex $v_1 \ni d(u, v_1) = k$, $d(v, v_1) = k$ where $k = \frac{n}{2}$.
Claim: S is the k-span of vertex v_1.
Proof is along similar lines as the earlier case and is omitted.

2.2 Properties of Cliques

Clique Distance - The distance $d(S_1, S_2)$ between two cliques S_1 and S_2 in G is defined to be the distance between clique vertex or clique edge of S_1 and the clique vertex or clique edge of S_2.

It may be noted that as $G \equiv T^n$, all cliques in G are either simultaneously centered about an edge or simultaneously centered around a vertex. Hence we do not have the case where we have to find the distance between an edge and a vertex.

Terminal edge - An edge $e = \{u, v\}$ is called a terminal edge if atleast one of $deg(u)$ and $deg(v)$ is 1.

kth order Terminal edges - The terminal edges of k-times end deleted tree of tree T are called kth order terminal edges of T.

Terminal Clique - A clique S is said to be a terminal clique if one of the following holds:
(i) n **is odd:** S is the k-span of edge $e = \{v_1, v_2\}$ where $k = \frac{n-1}{2}$ and either $v_1 \in L_k$ or $v_2 \in L_k$
(ii) n **is even:** S is the k-span of vertex v where $k = \frac{n}{2}$ and $v \in L_k$

Lemma 2.2 - *S is a terminal clique of graph G iff there exists a unique clique S' such that $\forall v \in S$, either $v \in S'$ or $v \notin$ any clique other than S. Further, the clique edge (or clique vertex) of S is adjacent to the clique edge (or clique vertex) of S'.*

Proof: Suppose S is a terminal clique.
Case I: n **is even**
$\therefore S$ is formed by a clique vertex v of G, $v \in L_k$, $k = \frac{n}{2}$.
Since $G \neq K_p$, $\therefore$ some neighbouring vertices of v are clique vertices.

Suppose A=$\{u_1, u_2, u_3 ... u_l\}$ is the set of adjacent vertices of v (which we call the adjacency of v).

Further suppose A_s=$\{u_{s_1}, u_{s_2}, u_{s_3}...u_{s_m}\}$ is the subset of A containing all the clique vertices.
Claim: $|A_s|$=1.
Assume, on the contrary, $|A_s| \geq 2$.
$\Rightarrow \exists u_{s_1}, u_{s_2} \in L_{k+i}$, $i \geq 0$.
$\therefore$ u_{s_1}, u_{s_2} are present in k-times end deleted tree of T,
$\Rightarrow v$ is an internal node
$\Rightarrow v \notin L_k$
This is a contradiction
$\therefore$ $|A_s|$=1.
Assume, without loss of generality A_s=$\{u_1\}$
Claim: The clique S_{u_1} formed by u_1 is the unique clique, S' that satisfies the conditions of the lemma.
Consider $w \in S \ni d(v, w) \leq k$ and $d(w, u_1) > k$
In that case, $d(w, v') > k$ $\forall$ clique vertices v'.
i.e., $w \notin$ any clique other than S.
Also, $\forall w \in S$, such that $d(v, w) \leq k$ and $d(w, u_1) \leq k$, $w \in S_{u_1}$
$\therefore$ S satisfies conditions of the lemma.
Case II: **n is odd**
The proof for this case is similar and is omitted.

Now we have to prove the converse part of the lemma.
Case I: **n is even**
Suppose S is a clique such that $\exists$ a unique clique S_u such that $\forall v \in S$, either $v \in S_u$ or $v \notin$ any clique other than S.
Say S_u = k-span of u and S = k-span of v.
Claim: u and v are adjacent in T.
Proof: Assume on the contrary that v' is a vertex which lies on the path between u and v in T.
Since $u, v \in L_{k+i}$ where $i \geq 0$, $\Rightarrow v' \in L_{k+i}$ where $i \geq 0$.
So, assume that S' is the clique formed by v'.
Consider $w \ni d(w, v) \leq k$ and $d(w, u) = k + 1$, then $d(w, v') \leq k$.
$\therefore$ $w \notin S_u$ and $w \in S'$, which is a contradiction.
Hence u and v are adjacent in T.
Claim: No other vertex adjacent to v is a clique vertex.
Proof: Say v_1 adjacent to v also forms a clique, S_{v_1}
$\Rightarrow v_1 \in L_k$
Now consider a vertex v_2 such that $d(v_2, v_1) = k - 1$ and $d(v_2, v) = k$ in T.
$\therefore$ $v_2 \in S$ and $v_2 \notin S_u$ but $v_2 \in S_{v_1}$, which is a contradiction.
Therefore no other vertex adjacent to v is a clique vertex.
$\therefore$ as a conclusion of above claims, $v \in L_k$.
$\Rightarrow S$ is a terminal clique.
Case II- n **is odd**

The proof for this case is similar to the one above and is omitted.

Definition-

Terminal vertex - A vertex $v \in V$ is said to be a terminal vertex if v is contained in exactly one clique S, and S is a terminal clique.

Lemma 2.3 - *If v is a terminal vertex in $G \equiv T^n$, then v is a leaf node in T.*

Proof:
Case I: n **is even**
By defintion of a terminal vertex, v is a vertex in G such that v is at a distance k from a clique vertex u in T. Also this clique vertex has only one clique vertex, say v_1 in its adjacency.
Specifically, $u \in L_k$
Suppose $v \notin L_0$.
$\Rightarrow$ k-times end-deleted tree of T has both v_1 and vertex adjacent to u on the path from u to v.
$\therefore$ u is not a leaf node in k-times end deleted tree of T.
$\therefore$ $u \notin L_k$, which is a contradiction
Case II: n **is odd**
The proof is by contradiction, similar to the one above.

It may be noted that the converse of Lemma 2.3 is not true in general, i.e., all leaf nodes in T need not be terminal vertices in T^n.

Lemma 2.4 - *If V_T=$\{v_1, v_2, v_3 \ldots v_k\}$ is the set of terminal vertices of a graph $G \equiv T^n$ and V_s is any subset of V_T then there exists a tree T_1 such that $G - V_s \equiv T_1^n$*

Proof: Assume $v \in V_s$
Then v is leaf node in T (by lemma 2.3).
Claim: $G - v \equiv (T - v)^n$.
This is a proved result (Gupta[2]).
It may be noted that $T - v$ is in fact a tree.
It is now easy to show, by induction, that for any $V_s \subseteq V_T$,
$G - V_s \equiv T_1^n$ where T_1 is a tree.

A graph G can have more than one tree as its nth root (Gupta[2]). But all such trees will have the same underlying structure. We continue this section in pursuit of that basic structure.

Definitions:

1st order terminal cliques - If $G \equiv T^n$ is a graph and V_T the set of its terminal vertices then the terminal cliques of $G' \equiv G - V_T$ are defined to be terminal cliques of 1st order.

kth order terminal cliques - Extending the previous definition to k-times terminal vertices deleted graph G, we get kth order terminal cliques of G.

Tree of cliques - A tree of cliques is that subgraph T' of T of which every vertex (or edge) forms a clique in T^n.

Lemma 2.5 - *The tree of cliques T' of T is the k-times end deleted tree of T where $k = \lfloor n/2 \rfloor$.*

Proof: This lemma follows as a direct consequence of Lemma 2.1.

Lemma 2.6 - **Intersection of Cliques** -*If G is a graph $\ni G \equiv T^n$ and S_1 and S_2 are cliques then the following property holds:*

$|S_1 \bigcap S_2| \geq n - r + 1$ where $d(S_1, S_2) = r$, $1 \leq r \leq n - 2$

$$|S_1 \bigcap S_2| = \begin{cases} 2 & \text{if } d(S_1, S_2) = n - 1 \\ 1 & \text{if } d(S_1, S_2) = n \\ 0 & \text{if } d(S_1, S_2) > n \end{cases}$$

Proof It may be noted that the size of intersection increases with the density of the tree (density is given by $|V|/\text{dia}(T)$).
So the intersection has less cardinality when T is sparse.
The most sparse tree possible is a staright path and this gives the minimum value of cardinality.
Consider a straight path of vertices

$v_0, v_1, v_2 \ldots v_n, v_{n+1} \ldots v_N$

Assume n is even.
Let $k = \frac{n}{2}$.
$\therefore S_{v_k} = \{v_0, v_1, v_2 \ldots v_n\}$
$S_{v_{k+1}} = \{v_1, v_2, v_3 \ldots v_{n+1}\}$
$S_{v_{k+2}} = \{v_2, v_3, v_4 \ldots v_{n+2}\}$
$S_{v_{k+3}} = \{v_3, v_4, v_5 \ldots v_{n+3}\}$
$\vdots$
$S_{v_{k+n}} = \{v_n, v_{n+1}, v_{n+2} \ldots v_{2n}\}$
$d(S_{v_k}, S_{v_{k+r}}) = r$, $1 \leq r \leq n - 2$
$S_{v_k} \bigcap S_{v_{k+r}} = \{v_r, v_{r+1}, v_{r+2} \ldots v_n\}$
i.e, $|S_{v_k} \bigcap S_{v_{k+r}}| = n - r + 1$
In particular
(i) if $r = n - 1$,
$S_{v_k} \bigcap S_{v_{k+r}} = \{v_{n-1}, v_n\}$
$\Rightarrow |S_{v_k} \bigcap S_{v_{k+r}}| = 2$
(ii) if $r = n$,
$S_{v_k} \bigcap S_{v_{k+r}} = \{v_n\}$
$\Rightarrow |S_{v_k} \bigcap S_{v_{k+r}}| = 1$
(iii) if $r > n$,

$S_{v_k} \bigcap S_{v_{k+r}} = \phi$
Hence, the proof.

The above lemma provides us with a distance ordering method for cliques. But unfortunately, it is of sparing use in sparse trees and of practically no use in dense trees.

The subsequent lemma gives us the basic structure of a tree T directly given $G \equiv T^n$. This more or less completes the hardwork we set out for.

Lemma 2.7 - *If $T = (V, E)$ is a tree and $G \equiv T^n$, then graph $G' = (V', E')$ is the $(n-1)$ times end-deleted tree of T, where*
V'=$\{v|v \in S_i \bigcap S_j$ *and* $|S_i \bigcap S_j|$=2$\}$
E'=$\{e|e \in S_i \bigcap S_j$ *and* $|S_i \bigcap S_j|$=2$\}$

Proof:
Case I: n **is even**
Suppose $k = \frac{n}{2}$. Say S_1 and S_2 are two cliques such that $|S_1 \bigcap S_2| = 2$. Also, say S_1 is the k-span of vertex u and S_2 is the k-span of vertex v.

Claim: $\forall v \in V'$, $v \in L_{n+i-1}$ where $i \geq 0$
Proof: By lemma 2.4, $d(S_1, S_2) = n - 1$.
$\therefore$ path from u to v can be written as:-
$u_1, u_2 \ldots u_{n-1}, u_n$
where $u_1 = u$ and $u_n = v$.
$S_1 \bigcap S_2$=$\{u_{\frac{n}{2}-1}, u_{\frac{n}{2}}\}$
Since $u, v \in L_{k+i}$, $i \geq 0$
$\Rightarrow u_{\frac{n}{2}}, u_{\frac{n}{2}-1} \in L_{k+i+\frac{n}{2}-1}$, $i \geq 0$.
$\Rightarrow u_{\frac{n}{2}}, u_{\frac{n}{2}-1} \in L_{n+i-1}$, $i \geq 0$
which proves the claim.

Claim: $\forall v \in L_{n-1+i}$, $i \geq 0$, $v \in V'$.
Proof: Say, $v \in L_{n-1+i}$
$\Rightarrow \exists u_1, u_2 \ni d(v, u_1) = k - 1$, $d(v, u_2) = k$ and $d(u_1, u_2) = n - 1$.
Let $S_{u_1} = k$-span of u_1 and S_{u_2}=k-span of u_2
$|S_{u_1} \bigcap S_{u_2}|$=2 and $v \in S_{u_1} \bigcap S_{u_2}$.
$\Rightarrow v \in G'$.

Claim: $\forall e \in E'$, $einT$
Proof: We have, from proof of first claim:-
$\forall e \ni e \in S_i \bigcap S_j$ and $|S_i \bigcap S_j| = 2$,
$e \in T$.

Claim: G' is connected.
Proof: Suppose,on the contrary, G' is not connected.
$\therefore \exists u, v$ such that $u, v \in G'$ but there is no path between u and v.
But as proved in the second claim, $\forall v \in L_{n-1+i}$, $v \in V'$.

$\therefore$ All vertices on the path between u, v are also in G' - which is a contradiction.
$\therefore G'$ is connected.
In third and fourth claims, it has been proved that G' is a subtree of T. Using first and second claims, it proves that G' is $(n-1)$ times end-deleted tree of T.
Hence the proof.
Case II: n **is odd**
The proof is similar and omitted.

3 Characterization of the n-th power of a tree, $G \equiv T^n$

Having proved the lemmas in section 2, we can now define a set of rules for a graph to be isomorphic to the nth power of some tree, T.
1) Let $S_1, S_2, S_3....S_n$ be the set of cliques of G.
Suppose $|S_1 \bigcap S_2| = c$
$\therefore d(S_1, S_2) \geq r = max\{n - c + 1, 0\}$;
i.e., there exist cliques $S_3, S_4, S_5...S_r$ such that

$d(S_1, S_3) = 1$;
$d(S_3, S_4) = 1$;
$\vdots$
$d(S_{r-1}, S_r) = 1$;
$d(S_r, S_2) = 1$;

2) If V=$\{v|v \in S_i \bigcap S_j$ and $|S_i \bigcap S_j| = 2\}$ and
E=$\{e|e \in S_i \bigcap S_j$ and $|S_i \bigcap S_j| = 2\}$
then (V, E) is a tree
3) $|S_i| \geq (n+1)\ \forall$ cliques S_i
4) Let $S_1, S_2, S_3...S_n$ be the set of cliques. Classify each of them as kth order terminal cliques
Our objective now is to generate the tree of cliques which satisfies conditions on clique distance and k-th order terminal cliques. We do it in the following manner:
(i) Find a terminal clique S.
(ii) Say S is formed by vertex v (or edge e).
(iii) Construct $G' = G - V'$ where V' is the set of vertices unique to S.
(iv) Recursively, find the tree of cliques, T' of G'.
(v) Say the clique S' (which satisfies the conditions of lemma 2.2 with respect to clique S) is formed by vertex v' (or edge e').
(vi) Append the vertex v (or edge e) to v' (or e').

(vii) The tree now formed is the tree of cliques of G.
The base case is provided by the tree found in the Condition 2 of above theorem.

Proof:
Claim: Conditions 1-4 are necessary.
Proof: Consider $G \equiv T^n$. By lemmas 1.1, 1.2, 1.3 and 2.1, G satisfies these conditions. Hence conditions 1-4 are necessary.
Claim: Conditions 1-4 are sufficient.
Proof: Assume that conditions are met.
By lemmas 2.2-2.7, we can easily show an isomorphism between G and the nth power of the generated tree T.
Hence, the conditions 1-4 are sufficient.
Hence the proof.

An example:- Consider the graph $G = (V, E)$ where V={1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16} E is given by the adjacency matrix shown below.

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}$$

Suppose we wish to find a tree T such that $G \equiv T^5$.
The cliques are:-
S_1={1,2,3,4,5,6,7,8,9,10}
S_2={2,5,6,7,8,9,10,11,12}
S_3={5,7,8,9,10,11,12,13,14}
S_4={7,8,9,10,11,12,13,14,15,16}
$\therefore$ The tree of cliques has 4 edges and S_1 and S_4 are terminal cliques
Suppose cliques S_1, S_2, S_3 and S_4 are formed by e_1, e_2, e_3 and e_4 respectively.
S_1, S_4 are 1st order terminal cliques.

$\therefore$ e_1 is a terminal edge in tree of cliques and is connected to e_2, and e_4 is a terminal edge in tree of cliques and is connected to e_3.
The 4-times end-deleted tree is determined to be a single vertex v.
On deleting the terminal vertices of S_1 (the set $\{1\}$) from G, we find a terminal clique S_4 in G.
We now delete its terminal vertices (the set $\{15,16\}$) from G.
Now G has only two cliques, S_2 centred at e_2 and S_3 centred at e_3.
$\therefore$ the tree of cliques has two edges.
$\Rightarrow$ the tree is P_3 (the only tree of two edges).
Clearly, the two edges are e_2 and e_3. Connect an edge e_1 to e_2 and an edge e_4 to e_3.
$\therefore$ The tree of cliques is P_5.

The complete tree can be deduced from the tree of cliques which is 2-times end deleted tree of original tree.

The complete tree can be obtained heuristically by applying following methodology:-
We see that terminal vertex of S_1 is 1.
$\therefore$ vertex 1 is at a distance of 2 from edge e_1 and a distance of 3 from edge e_2.
Similarly vertex 16 is at a distance of 2 from edge e_4 and a distance of 3 from e_3.
The rest of the tree is computed in a similar manner.

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