

# HW 1 Solution Outline

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## Question 1

(Study/Review of the principle of mathematical induction) (PMI). Prove the following using PMI:

$$1^3 + 2^3 + 3^3 + \dots + n^3 = [n(n+1)/2]^2$$

## Answer

**Base Case (n = 1):** So, to prove the equation using PMI, we first prove the base case, for  $n = 1$ . For  $n = 1$ , the left hand side (lhs) = 1, and right hand side (rhs) = 1. Thus, the base case holds.

**Induction Hypothesis (IH):** Let us assume that the equation holds for all values of  $n$ , up to and including  $N$ . Thus, we can say that:

$$1^3 + 2^3 + 3^3 + \dots + N^3 = [N(N+1)/2]^2$$

**Induction Step (IS):** To prove the induction step, we wish to prove that the equality also holds for  $N + 1$ .

Using IH, we can say that:

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \dots + N^3 + (N+1)^3 &= [N(N+1)/2]^2 + (N+1)^3 \\ &= [N^4/4 + N^3/2 + N^2/4] + [N^3 + 3N^2 + 3N + 1] \\ &= N^4/4 + 3N^3/2 + 13N^2/4 + 3N + 1 \\ &= \frac{1}{4}(N^4 + 6N^3 + 13N^2 + 12N + 4) \\ &= (N^2 + 2N + 1)(N^2 + 4N + 4) / 4 \\ &= (N+1)^2(N+2)^2/4 \end{aligned}$$

Thus, the equation also holds for  $N+1$ .

Hence, by Principle of Mathematical Induction, the equation holds for all  $n \geq 1$ . (Note that 1 is the value we had proved for in Base Case.)

## Solution 2

Asymptotic Big O Notation

(a)  $4n^3 + 2n^2 + 17n + 40 = O(n^3)$

$n_0 = 1, k = 63$ . The equation that we want to prove in that case is that:

$$\forall n \geq 1, 4n^3 + 2n^2 + 17n + 40 \leq 63n^3$$

This is true because  $\forall n \geq 1$ :

$$4n^3 \leq 4n^3$$

$$2n^2 \leq 2n^3$$

$$17n \leq 17n^3$$

$$40 \leq 40n^3$$

Thus, adding all of them up, we get that  $4n^3 + 2n^2 + 17n + 40 \leq 63n^3$

- (b) To prove this, first prove two intermediate results that the first series sums to  $n(n+1)/2$ , and the second series sums to  $n(n+1)(2n+1)/6$ . One example values of  $n_0$  and  $k$  are:  $n_0 = 7, k = \frac{1}{2}$ . One way to arrive at these values is to be simply pick high values of  $n_0$  and  $k$ , such as  $n_0 = 10, k =$

1000. Those answers are entirely fine (though a bit defensive). But if you want some smart values, you want those values to be as small as possible. In that case, we can pick a value of  $k$ , such as  $k = \frac{1}{2}$ , and then to find the best possible value of  $n_0$ , by doing the following math:

$$n^3/3 + n^2 + 2n/3 \leq n^3/2$$

$$n^2 + 2n/3 \leq n^3/6$$

$$n + 2/3 \leq n^2/6$$

$$6n + 4 \leq n^2$$

$$n^2 - 6n - 4 \geq 0$$

$$(n - 3)^2 \geq 13$$

$$n - 3 \geq 4$$

$$n \geq 7$$

We also observe that the value of  $k$  could not have been less than  $1/3$ , since that is the coefficient of  $n^3$  in the left hand side. Indeed, if the value of  $k$  is chosen as  $1$ , instead of  $\frac{1}{2}$ , then we can arrive at the value of  $n_0$  to  $2$  using the following calculation.

$$n^3/3 + n^2 + 2n/3 \leq n^3$$

$$n^2 + 2n/3 \leq 2n^3/3$$

$$n + 2/3 \leq 2n^2/3$$

$$3n + 2 \leq 2n^2$$

$$n^2 - 3/2n - 1 \geq 0$$

$$(n - 3/4)^2 \geq 25/16$$

$$n - 3/4 \geq 5/4$$

$$n \geq 2$$

To visualize these functions, consider the following Excel workbook.

- (c) To see this result, we note that each term is no larger than  $\log(n)$ , and there are  $n$  terms. Thus, the sum of these terms must be no larger than  $n \log(n)$ . This sum results for  $n_0 = 1$ ,  $k = 1$ .

It is much more interesting to prove however, that the right hand side is also a big O of the left hand side, that is,  $n \log(n) = O(\log(1) + \log(2) + \dots + \log(n))$ .

- (d) Note that  $\log(n^2) = 2 \log(n)$

### Question 3

Claim:  $n-1$  comparisons are necessary to find the largest number given  $n$  numbers.

#### Solution

Consider a tennis tournament. In each match, at most one (in fact exactly one) person loses. When we have a winner, everyone else must have lost at least once (otherwise they would be in contention to be the winner). Thus, if we have to have a winner,  $n-1$  people must have lost at least once, thus  $n-1$  matches must have already taken place.

### Question 4

Give an efficient algorithm to find both the maximum and minimum of the numbers, given  $n$  numbers.

#### Solution

First conduct  $n/2$  comparisons to compare adjacent numbers, such as  $a[0]$  and  $a[1]$  etc. Put the larger of the numbers in one pile, and the smaller of the numbers in the other pile. Now, conduct  $n/2 - 1$

comparisons (as in Question 3) to find the largest number noting that it must be in the pile of large numbers. Conduct  $n/2 - 1$  comparisons to find the smallest number noting that it must be in the pile of small numbers. Total number of comparisons is  $3n/2 - 2$ . So, given 100 numbers, it takes 148 comparisons.